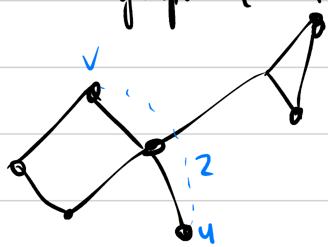
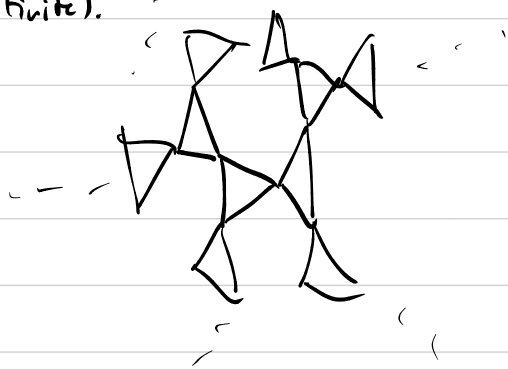
Metric Spaces and Topology

Lecture 2

Examples (continued). ○ Let $G := (\overset{\text{vertices}}{V}, \overset{\text{edges}}{E})$ be a connected undirected graph (finite or infinite).





Let $d : V \times V \to \mathbb{N}$

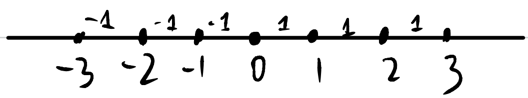
$(u,v) \mapsto$ the length of the shortest path from u to v.

Check that this is indeed a metric.

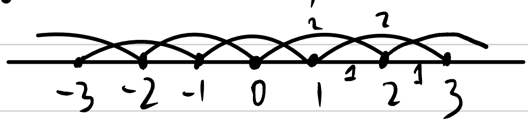
□ For example, let $n \in \mathbb{N}^+$ and $V := \{0,1\}^n :=$ the set of all 0-1 sequences of length n, e.g. $\{0,1\}^2 = \{00, 01, 10, 11\}$. We put an edge between two vertices $u, v \in \{0,1\}^n$ if u and v differ by 1 bit, e.g. 011 and 010. This defines a graph called the Hamming graph, denoted H. Thus, there is the shortest path metric on this graph. Show that

$d_H(u,v) = \#$ of bits by which u and v differ.
This is also called the Hamming distance.

○ <u>Groups as metric spaces.</u> Let Γ be a group (typically infinite). Let S be a symmetric set of generators for Γ (symmetric := closed under inverses). E.g. $\mathbb{Z}$ with $S := \{\pm 1\}$. The Cayley graph of Γ with respect to S is $\mathrm{Cay}_S(\Gamma) := (\Gamma, E_S)$, where we put an edge between $\gamma_1, \gamma_2 \in \Gamma$ if $\gamma_1 \cdot \sigma = \gamma_2$ for some $\sigma \in S$.

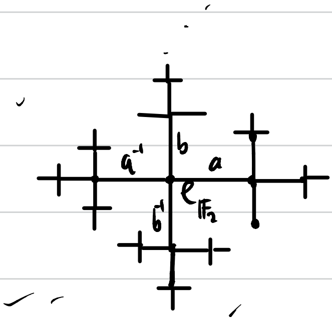


E.g. $\mathbb{Z}$, $S := \{\pm 1, \pm 2\}$.



E.g. $\mathbb{F}_2 := \langle a, b \rangle$ free gp on 2 generators.
" $\{$reduced words in symbols $a^{\pm 1}, b^{\pm 1}\}$,
where reduced means a and a^{-1}, b and b^{-1} don't appear next to each other. E.g. $a b^{-1} b^{-1} a b$ is reduced but $a b b^{-1} b^{-1}$ isn't. The group

operation $w_1 \cdot w_2$ is defined by the concatenating the words w_1 and w_2 into $w_1 w_2$ and reducing it (i.e. cancelling the neighbouring a, a^{-1} and b, b^{-1}). Let $S := \{a^{\pm 1}, b^{\pm 1}\}$.

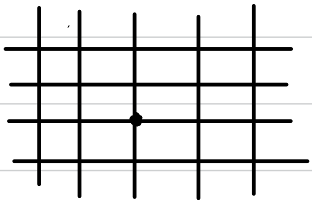


4-regular tree

Viewing Cayley graphs as metric spaces (with the shortest path metric) started a very active math area (by Gromov) called geometric group theory. This subject studies the geometric properties of groups and see how they correlate with algebraic properties.

☐ For example, the growth of the group.
For Γ a gp and S a symmetric gen. set, let $g: \mathbb{N} \to \mathbb{N}$ be defined by $g(n) :=$ # of

elements in the ball of radius n at the identity in $Cay_S(\Gamma)$. E.g. For $\mathbb{Z}$ & $S:=\{\pm 1\}$, $g(n)=2n+1$.
For $\mathbb{F}_2$ with $S:=\{a^{\pm 1}, b^{\pm 1}\}$, $g(n) = 4\cdot 3^{n-1}+1$.
For $\mathbb{Z}^2$ with $S:=\{(\pm 1,0),(0,\pm 1)\}$, $g(n) = C\cdot n^2 + D$.



The "asymptotic" behavior of g doesn't depend on S, e.g. polynomial stays polynomial.

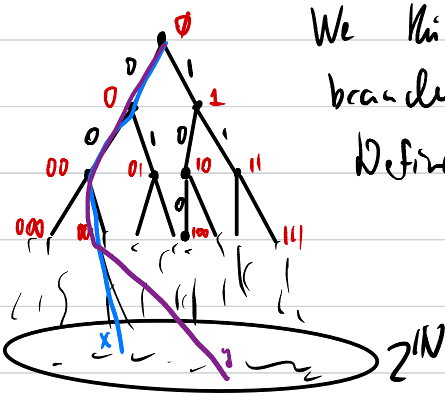
Question. Which finitely generated groups have polynomial growth?

It's not terribly hard to show that all f.g. abelian groups & even nilpotent groups & even virtually nilpotent groups have polynomial growth.

Gromov's thm. A finitely gen. group has poly. growth if and only if it is virtually nilpotent.

This is a stunning theorem as it recovers an algebraic / qualitative property from a geometric/quantitative one.

- Cantor space. $X := \{0,1\}^{\mathbb{N}} =: 2^{\mathbb{N}} := \{(x_n)_{n\in\mathbb{N}} : x_n \in \{0,1\}, \forall n \in \mathbb{N}\}$.



We think of $2^{\mathbb{N}}$ as the set of infinite branches through the complete binary tree.

Define $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to [0,1]$ by

$$(x,y) \mapsto 2^{-n(x,y)},$$

$n(x,y) :=$ the least index i with $x(i) \neq y(i)$.

Check that in fact d is an ultra-metric, i.e.

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

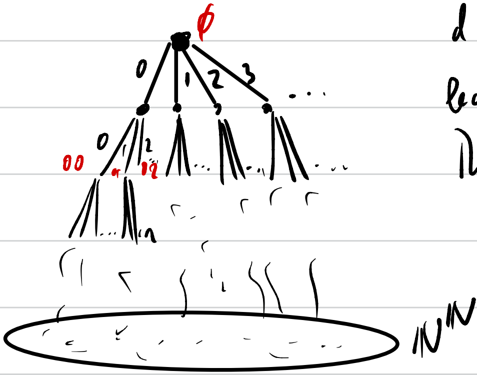
Let $x \in 2^{\mathbb{N}}$, what is $B_{2^{-n}}(x)$? $B_{2^{-n}}(x) = \{y \in 2^{\mathbb{N}} : y|_n = x|_n\}$

$$\| \quad\quad =: [x_0 x_1 x_2 \dots x_{n-1}]$$

$$\overline{B}_{2^{-(n-1)}}(x)$$

- Baire space.

Let $X := \mathbb{N}^{\mathbb{N}} := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{N} \text{ for all } n \in \mathbb{N}\}$.



$d(x,y) := 2^{-n(x,y)}$, where $n(x,y) :=$ the least index i s.t. $x|_i = y|_i$.

This is again a ultra-metric.

Def. Let (X,d) be a metric space. For a nonempty set $S \subseteq X$, define its *diameter* by $\mathrm{diam}_d(S) := \sup_{x,y\in S} d(x,y)$.

Def. Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f: X \to Y$ is called an *isometry* if $\forall x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1,x_2)$.

Note that isometry is necessarily an injection, but doesn't have to be surjective.

Find a metric d on $\mathbb{R}^n$ such that the map $f: \{0,1\}^n \to \mathbb{R}^n$

given by $101101 \mapsto (1,0,1,1,0,1)$ is an isometry from the Hamming distance to d.